

Trace Formulas and Bogomolny's Transfer Operator

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The trace formulas commonly used in discussing spectral properties of quantum or wave systems are derived simply and directly from the Bogomolny transfer operator. Special cases are the Gutzwiller formula, the Berry-Tabor formula, and the perturbed Berry-Tabor formula. A comment is made about interpolation formulas proposed in the latter case.

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I. INTRODUCTION

The Gutzwiller trace formula¹ and its relatives are the cornerstone of much of the subject of 'quantum chaology'. This formula gives a formal quasiclassical expression for the density of states, in other words, for the energy levels, of a 'hard' chaotic system. The Berry-Tabor trace formula^{2,3} is the corresponding formula for an integrable system. These are descendants of the exact Selberg⁴ trace formula, which applies to spaces of negative curvature.

Twenty-five years of work on the trace formulas leads to the conclusion that they are not especially useful for finding actual energy levels. This is because they are divergent formulas unless they are smoothed over energy. On the other hand, many interesting observables can be expressed in terms of the density of states smoothed over some energy scale. For example, the specific heat or magnetic susceptibility are observables of this type. The smoothing can come about because of finite temperature or finite experimental resolution, say. With appropriate smoothing, the trace formulas are very useful.

We here provide a derivation of these and related formulas based on Bogomolny's transfer operator⁵. One of the main virtues of the transfer operator is that it can be used to study energy levels, wavefunctions and scattering amplitudes. The possibility of obtaining the trace formulas from the transfer operator is widely known, and certainly was to Bogomolny, especially in the Gutzwiller case.

An explicit derivation of the Berry-Tabor formula or of the perturbed Berry-Tabor formula that is based on the Bogomolny operator does not seem to have appeared in print, however. That is the main purpose of this paper. We restrict consideration to the two-dimensional case, which is by far the most important in practice, and to a closed system with a discrete spectrum.

The original derivation of the Gutzwiller formula¹ was based on a stationary phase evaluation of Feynman's path integral expression for the quantum propagator. This reduces the propagator from a sum over all paths to a sum over just the classical paths. Although the Feynman formulation gives a persuasive picture of the relationship of quantum to classical mechanics, it does have some defects. For one thing, it is mathematically not very well defined. Also, its perspective on classical mechanics is very elementary. In particular, it does not incorporate the insights gained by use of Poincaré's surface of section method. The present derivation removes these defects.

The Berry-Tabor formula was derived in several ways^{6,2,3}. For example, the density of states expressed as a sum over explicitly known energy levels can be transformed by use of the Poisson sum formula. Action-angle variables are also useful to obtain this formula².

With considerable generality, for a given problem, that is, for a given Hamiltonian, there exists⁷ a mathematically well defined kernel or operator $K(q, q', E)$, whose arguments q, q' are on a one-dimensional *Poincaré surface of section* [SS]. The exact spectrum is determined by the Fredholm integral equation

$$\psi(q) = \int_{SS} dq' K(q, q', E) \psi(q'), \quad (1)$$

which has nontrivial solutions only if E is on the spectrum. The integral runs over the surface of section. This is an exact quantum version of the classical SS method. A well known and much used example⁸ is a billiard where the surface of section is the boundary and $K(q, q', E)$ is a Hankel function with argument $kL(q, q')$. Here $k = \sqrt{2mE}/\hbar$ with m the mass and E the energy of the particle in the billiard, and L is the chord distance between points on the boundary specified by the q 's. The exact or numerically approximated method is in this case called the *boundary integral method*⁵.

II. BOGOMOLNY'S TRANSFER OPERATOR, T

In more general cases, K does not have a simple expression. However, Bogomolny⁵ in effect argued that in quasiclassical approximation K may be replaced by its asymptotic form for large separation of points q, q' . In this approximation, the operator, called T , is quite simple, because it involves only a few, often just one, short orbits. Namely,

$$T(q, q', E) = \left(\frac{1}{2\pi i \hbar} \left| \frac{\partial^2 S(q, q', E)}{\partial q \partial q'} \right| \right)^{\frac{1}{2}} \times \exp \left[\frac{i}{\hbar} S(q, q', E) + \frac{i\pi}{2} \nu \right], \quad (2)$$

where S is the action $\int pdq$ along the classical path of energy E , which leaves the surface of section at q' and returns to it *for the first time* at point q .

Care must be taken with the phase attributed to the prefactor and with phase changes encountered at points

along the path where the leading order semiclassical approximation fails, e.g. at a billiard boundary. These are incorporated into the ‘Maslov’ phase ν . It can be regarded as part of the action S , i.e. $S \rightarrow S + h\nu/4$ and thus is a quantum correction. Also, in doing integrals such as those below, contributions to ν may be obtained. We shall, however, suppress ν in the sequel, since that is not the point of this work.

It should be noted that S is the *generator of the classical surface of section map*, namely $p = \partial S(q, q')/\partial q$, $p' = -\partial S/\partial q'$, where p, p' are the momenta of the orbit parallel to the surface of section at q, q' . S thus contains *all the long-time and long-orbit information* of the classical mechanics.

III. THE GENERAL QUASICLASSICAL TRACE FORMULA

Eq. (1), with K replaced by T , has a solution when the Fredholm determinant⁹

$$D(E) = \det[1 - T(E)] = 0. \quad (3)$$

The density of states is given by the logarithmic derivative^{5,8}

$$d(E) - \bar{d}(E) \equiv d_{osc}(E) = \frac{-1}{\pi} \text{Im} \left[\frac{d \ln D(E + i\epsilon)}{dE} \right], \quad (4)$$

where $d(E) = \sum_a \delta(E - E_a)$ and $\bar{d}(E)$ is the smoothed (Weyl) density of states. Using the relationship $\ln \det(1 - T) = \text{Tr} \ln(1 - T)$, and expanding the logarithm as if T were small we have

$$d_{osc}(E) = \frac{1}{\pi} \text{Im} \sum_{n=1}^{\infty} \frac{1}{n} \frac{d\tau_n(E)}{dE}, \quad (5)$$

where

$$\tau_n(E) = \text{Tr} T(E)^n. \quad (6)$$

Since T is in a quasiclassical sense close to a unitary operator, some of the eigenvalues of T are close to the unit circle, which makes the expansion (5) divergent for real E . However, in this paper we ignore this problem, since, as we said, in practice the trace formulas are useful only when averages are taken, which suppress the contribution of the terms with large n .

The trace formulas are obtained by evaluating the traces defining the τ_n 's in stationary phase approximation, if this is meaningful. There are n integrals to be carried out, i.e.

$$\tau_n = \int \dots \int dq_1 \dots dq_n T(q_1, q_2) \dots T(q_n, q_1). \quad (7)$$

The fundamental composition relation

$$T_2(q, q') = \int d\bar{q} T(q, \bar{q}) T(\bar{q}, q') \quad (8)$$

evaluated in stationary phase is given by⁵

$$T_2(q, q') = \sum_p \left(\frac{1}{2\pi i \hbar} \left| \frac{\partial^2 S_{2p}(q, q', E)}{\partial q \partial q'} \right| \right)^{\frac{1}{2}} \times \exp \left[\frac{i}{\hbar} S_{2p}(q, q', E) \right], \quad (9)$$

where

$$S_{2p}(q, q', E) = S(q, q_p) + S(q_p, q') \quad (10)$$

is the action of a classical orbit from q' to q at energy E , which arrives at the surface of section at q for the *second* time after having crossed the *first* time at q_p . The stationary phase condition is $\partial[S(q, q_p) + S(q_p, q')]/\partial q_p = 0$. The solution of this equation, q_p , is a function of q, q' . There are, in general, more than one solution q_p to this stationary phase condition, which can lead to an exponential proliferation of orbits as a function of n . Thus, a T operator corresponding to multiple SS crossings can be defined, and it has exactly the same form as Eq. (2), although there are in general many terms of this type.

There are two common situations. In the first, all n integrals may be performed by stationary phase. This is the Gutzwiller assumption. In the second case, the first $n-1$ integrals are well approximated by stationary phase but this method fails for the last integral.

Eq. (9) may be iterated to obtain the final integral

$$\tau_n = \int dq \sum_p \left(\frac{1}{2\pi i \hbar} \left| \frac{\partial^2 S_p(q, q', E)}{\partial q \partial q'} \right| \right)^{\frac{1}{2}} \times \exp \left[\frac{i}{\hbar} S_p(q, q, E) \right]. \quad (11)$$

Here p denotes a ‘closed’ orbit from q , which crosses the SS $n-1$ times and arrives back at q at its n th encounter with the SS.

IV. THE GUTZWILLER TRACE FORMULA

In the Gutzwiller case, the final q integral may be done by stationary phase also, that is, the phase $S_p(q, q, E)/\hbar$ is a rapidly varying function of q . This picks out as stationary points the solutions of $\partial S_p(q, q', E)/\partial q + \partial S_p(q, q', E)/\partial q' = p - p' = 0$ at $q = q' = q^*$. Thus periodic orbits are selected. Let q, q' be in the vicinity of q^* and expand

$$S_p(q, q') \simeq S_p(E) + p^*(\delta q - \delta q') + \frac{1}{2} W_{11} \delta q^2 + W_{12} \delta q \delta q' + \frac{1}{2} W_{22} \delta q'^2, \quad (12)$$

where $\delta q = q - q^*$, $\delta q' = q' - q^*$ and p^* is the momentum of the periodic orbit. The W matrix depends on energy and on the orbit. Let the orbit p of ‘length’ n (i.e. returning to the surface of section n times) consist of r repetitions of ‘length’ s , with $n = rs$. The q integral is then done to obtain the contribution of this periodic orbit to the trace

$$\tau_p = s \left| \frac{W_{12}}{W_{11} + 2W_{12} + W_{22}} \right|^{\frac{1}{2}} \exp \left[\frac{i}{\hbar} S_p(E) \right]. \quad (13)$$

The factor s arises because the stationary point for the last integral can occur (or, in other words, the starting/ending point for the periodic orbit can be chosen) at any of s surface of section crossings.

The prefactor is usually expressed in terms of the monodromy matrix M_p of orbit p , which connects the final momentum-position $p - p^* = \delta p = W_{11}\delta q + W_{12}\delta q'$ and δq to the initial $\delta p' = -W_{12}\delta q - W_{22}\delta q'$ and $\delta q'$. That is

$$M_p = \begin{pmatrix} -\frac{W_{11}}{W_{12}} & \frac{W_{12}^2 - W_{11}W_{22}}{W_{12}} \\ \frac{1}{-W_{12}} & -\frac{W_{22}}{W_{12}} \end{pmatrix}, \quad (14)$$

so that the prefactor in Eq. (13) can be written $|\det(M_p - 1)|^{-1/2}$.

In the case of a repeated orbit, $S_p = rS_s$, and $M_p = M_s^r$. In taking the derivative of τ_p only the rapidly varying phase needs to be differentiated, giving a factor of $dS_s(E)/dE = T_s$, the period of the primitive orbit, leading to the final expression

$$d_{osc} = \sum_{r,s} \frac{T_s/\hbar}{|\det(M_s^r - 1)|^{1/2}} \cos \left\{ r \left[\frac{S_s(E)}{\hbar} + \frac{\pi}{2} \nu_s \right] \right\}, \quad (15)$$

in which we restore the Maslov index.

V. THE BERRY-TABOR FORMULA

The above formula does not apply to integrable systems. In that case, the periodic orbits are not isolated. The eigenvalues λ of the monodromy matrix are, for isolated orbits, $\lambda = e^{\pm i\gamma}$, where for an unstable orbit the Lyapunov exponent $\gamma > 0$, and for a stable orbit γ is pure imaginary. For an integrable system, γ vanishes and formula (15) is infinite.

In the case of an integrable system, it is easiest to go to action-angle variables², θ, I, Θ, J using as surface of section $\Theta = 0$, and energy $E = H(I, J)$ fixed. Then the action of the T operator becomes $S(q, q') \rightarrow S(\theta - \theta')$. The replacement of the action depending on two variables separately by a function of the difference of coordinates only characterizes the integrable system. In fact, the action is

$$S(\theta - \theta', E) = (\theta - \theta')I + 2\pi J. \quad (16)$$

The actions I, J specify the invariant torus, on which the orbit remains. The frequencies of rotation around this torus are $\omega_I = \partial H/\partial I$ and $\omega_J = \partial H/\partial J$. We may equally well use energy E and winding number $\omega \equiv \omega_I/\omega_J = (\theta - \theta')/2\pi$ as variables to specify the orbit. [This fails for harmonic oscillators where the frequencies are constants. Modifications are necessary in that case².] Thus, I, J are regarded as functions of $E, \theta - \theta'$. It is easy to see that $\partial S/\partial \theta = I$. Obviously, the eigenfunction of the T operator is in this case $\psi = \exp iI\theta$.

The first $n - 1$ integrals may be carried out by stationary phase, as before, giving a result $S_p(\theta - \theta')$ where θ , which is physically equivalent to θ' , is set equal to $\theta' + 2\pi m$. The winding number is m/n , i.e. it is rational. Clearly $S_p(\theta - \theta') = nS(\Delta\theta)$, where $\Delta\theta = (\theta - \theta')/n$. Remark first that in this limit p is a periodic orbit. It is a periodic orbit for each θ with identical period and other properties, in other words, this describes a set of periodic orbits, which are not isolated.

The final integral Eq. (11) is then trivial, giving a factor 2π . Note a factor $i^{-\frac{1}{2}}$, which remains and gives a shift $\pi/4$ in the final answer below. The prefactor involves the second derivative (at constant E), $S_p'' = \frac{1}{\pi} \partial I / \partial \Delta\theta$. This has traditionally been expressed in terms of the function $g_E(I) = J$, which determines the action J given the energy and I . One finds that $g_E' = -\Delta\theta/2\pi$, the derivative being taken at fixed E . Therefore $S_p'' = -(2\pi n g_E'')^{-1}$. Putting this all together we obtain the standard result^{2,3}

$$d_{osc} = \sum_p \frac{T_p}{\pi \hbar^{3/2} n^{3/2} |g_E''|^{1/2}} \cos \left(\frac{S_p}{\hbar} + \frac{\pi}{2} \nu_p - \frac{\pi}{4} \right). \quad (17)$$

There are other cases, in which the last integral Eq. (11) cannot be done by stationary phase. There may be particular sets of nonisolated orbits in an otherwise chaotic system. Prominent examples are the bouncing ball orbits in the stadium¹⁰ or Sinai billiard¹¹. There are also nonisolated orbits in the ray splitting billiard¹².

VI. PERTURBED BERRY-TABOR FORMULA

Another important example is the perturbed integrable system. Then the action defining the T operator has the form $S(\theta - \theta') + \epsilon W_1(\theta, \theta')$, and $\epsilon \ll 1$. To the leading order in ϵ , we may assume that the stationary phase points of the first $n - 1$ integrals are not shifted. The last integral then involves

$$I_W = \frac{1}{2\pi} \int d\theta \exp \left[\frac{i\epsilon}{\hbar} \hat{W}(\theta) \right], \quad (18)$$

where $\hat{W}(\theta) = \sum_{r=1}^n W_1(\theta - \theta_{r-1}, \theta - \theta_r)$, and $\theta_r = 2\pi m r/n$ for the winding number m/n . It can be shown that $\hat{W}(\theta) = W_n(\theta, \theta - 2\pi m)$, where $W_n(\theta, \theta')$ is the first order correction to the action for the orbit that returns to the SS for the n th time. The perturbed result is just the Berry-Tabor result with the substitution $\cos \phi \rightarrow \text{Re}[I_W \exp(i\phi)]$, where ϕ is the argument of cosine in Eq. (17)¹³.

Clearly, I_W is a sort of generalized Bessel function. Formulas for I_W interpolating between $\epsilon = 0$ and ϵ/\hbar large have been given in the literature. The first were based on the idea that often \hat{W} will be well approximated by the first terms of its Fourier series¹³, e.g. $\hat{W} \approx w_0 + w_1 \cos(\theta - \theta_0)$. The phase shift θ_0 is of no importance, and $I_W \approx \exp(i\epsilon w_0/\hbar) J_0(\epsilon w_1/\hbar)$, where J_0 is the Bessel function.

Interpolation formulas along this line using more parameters have been given in the literature¹⁴. We give an alternative four parameter formula in the Appendix, since we believe the existing formulas are defective.

One comment concerning these formulas based on a small parameter ϵ is in order. Both *classical* perturbation theory and *quantum* results based on small ϵ have relatively large effects, proportional to $\sqrt{\epsilon}$, when the system is in the neighborhood of a resonance. Furthermore, the trace formulas apply precisely to the resonant cases.

Nevertheless, it is the case that for the perturbed trace formula, the corrections incorporated into Eq. (18) can be expanded in powers of ϵ , and such a series is convergent.

The reason is as follows: To find energy levels and wave functions, one must solve Eqs. (3,1). This, in effect, depends on the very high terms and convergence properties of the trace formulas. The $\sqrt{\epsilon}$ appears if one is interested in this level of accuracy¹⁵. However, to look at long range energy level correlations, and the smoothed density of states, it is only necessary to consider the leading terms in the trace formula expansion. These leading terms can be calculated from short periodic orbits. Already classically, it is known that short time quantities can be calculated perturbatively as a power series in ϵ .

VII. CONCLUSION

The Bogomolny transfer operator T provides a unified approach to the quasiclassical description of wave systems, without needing to make specific reference to the chaotic or integrable properties of the system. Its greatest importance is, no doubt, that it provides a method to find actual energy levels and wave functions. However, it can also be used to find the trace formulas, which are widely used in this field. This derivation of the trace formulas is quite easy and direct and it is on at least as firm a mathematical footing as most other derivations.

APPENDIX A: UNIFORM APPROXIMATION

Suppose $\hat{W}(\theta)$ has a single maximum and minimum at $\theta = \theta_x, \theta_n$, respectively. For large ϵ/\hbar the integral (18) can be done by stationary phase. This is sometimes called the Gutzwiller case because the continuum of identical periodic orbits has turned into two isolated orbits. The integral in this approximation depends on four parameters, namely, the magnitudes of \hat{W} and \hat{W}'' at the extrema. It does not depend directly on the position of the extrema. Thus, in order to have the correct large ϵ/\hbar limit, one should use a four parameter expression for I_W .

Inspired by Ref.¹⁴, [UGT], we parametrize $\hat{W}(\theta) = W^{(0)} + W^{(1)} \cos[\xi(\theta)]$, where $\xi(\theta_x) = 0$ and $\xi(\theta_n) = \pi$. Note that $W^{(0)}$ and $W^{(1)}$ are not necessarily equal to w_0, w_1 . We may as well take $\theta_x = 0$ and $\theta_n = \pi$, to simplify things. Now take ξ as the solution of $\theta = \xi - A \sin \xi - B \sin 2\xi$. This goes beyond UGT, who take $B = 0$. The derivative is $\hat{W}' = -\xi'(\theta)W^{(1)} \sin \xi$, and the

second derivative, evaluated at an extremum, is $\hat{W}'' = -(\xi')^2 W^{(1)} \cos \xi$. Since $d\theta/d\xi = 1 - A \cos \xi - 2B \cos 2\xi$, we find $\xi'(0) = [1 - A - 2B]^{-1}$ and $\xi'(\pi) = [1 + A - 2B]^{-1}$. We may solve for $A, B, W^{(0)}$, and $W^{(1)}$ to obtain a parametrized version of \hat{W} , which interpolates between $\epsilon/\hbar = 0$ and ϵ/\hbar large.

The integral Eq. (18) now is

$$I_W = \frac{I_0}{2\pi} \int d\xi (1 - A \cos \xi - 2B \cos 2\xi) \times \exp \left[i(\epsilon/\hbar) W^{(1)} \cos \xi \right], \quad (A1)$$

where $I_0 = \exp [i(\epsilon/\hbar) W^{(0)}]$. The integrals may be expressed as Bessel functions J_m , whose arguments are $\epsilon W^{(1)}/\hbar$, giving

$$I_W = I_0 (J_0 - iA J_1 + 2B J_2). \quad (A2)$$

As an exercise, and to compare with UGT, we calculate I_W for coupled quartic oscillators. The Hamiltonian for this case is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + ax^4 + by^4 + \epsilon x^2 y^2. \quad (A3)$$

The correspondence with UGT goes as follows: $a \leftrightarrow a(\lambda)/b$, $b \leftrightarrow a(\lambda)b$, $\epsilon \leftrightarrow 2\lambda a(\lambda)$, where $a(\lambda)$ is a specified function of λ . However, we may choose the unit of length such that $a(\lambda) = 1$, i.e. $a = 1/b$. Thus λ , the small parameter of UGP, is in effect our $\epsilon/2$.

Using action-angle variables, with $\Delta\Theta = 2\pi$, we find

$$S(\theta - \theta') = \frac{4KE^{3/4}}{3\pi} \left[\frac{(\theta - \theta')^4}{a} + \frac{(2\pi)^4}{b} \right]^{1/4}, \quad (A4)$$

where $K \equiv K(m = 1/2)$ is the complete elliptic integral of the first kind¹⁶. Here E is the energy of the unperturbed system. The perturbing action W_n is found (to leading order in ϵ) in terms of Jacobi elliptic functions¹⁶ as

$$W_n(\theta, \theta') = -\frac{KE^{3/4}}{4\pi(ab)^{1/4}} \frac{\omega}{[a + b\omega^4]^{3/4}} \times \int_{-2\pi n\omega}^0 d\theta'' \text{sd}^2[\alpha(\theta'' + \theta)] \text{sd}^2[\alpha\theta''/\omega], \quad (A5)$$

where $\alpha = 2K/\pi$ and $\omega = (\theta - \theta')/2\pi n$. We focus on the 1-1 resonance, $\omega = 1$, and $\hat{W}(\theta) = W_1(\theta, \theta - 2\pi)$. Note that the shape of $\hat{W}(\theta)$ for fixed ω is independent of a and E and depends only on the winding number.

Fig. 1 gives a graph of $\hat{W}(\theta)$. It has two maxima and minima, which are symmetrically related. It is easy to extend the interpolation expression to take this into account. $\text{Re}[I_W]$ as a function of ϵ/\hbar is presented in Fig. 2, and $\text{Im}[I_W]$ is shown in Fig. 3. We find that the interpolation formula and the exact integral agree to within a part in 10^{-5} (inset Fig. 2).

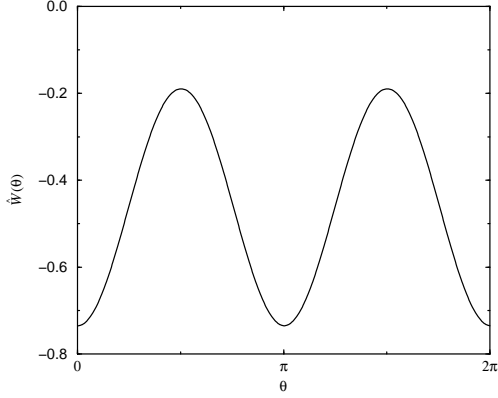


FIG. 1. The function $\hat{W}(\theta)$ vs. θ for anharmonic oscillators with $x^2 y^2$ coupling, for the (1,1) resonance. The parameters a , b , and E are defined in the text and each is taken to be unity.

UGT assume B is zero. This means that they cannot generally satisfy the asymptotic condition for large ϵ/\hbar . It can be seen that $B = 0$ implies a relation between the curvatures at minimum and maximum, namely,

$$\sqrt{W^{(1)}/\hat{W}''_{\min}} + \sqrt{W^{(1)}/|\hat{W}''_{\max}|} = 2/r, \quad (\text{A6})$$

where r is the number of maxima (or minima) of $\hat{W}(\theta)$ per 2π ($r = 2$ for 1-1 resonance). However, for quartic oscillators with $x^2 y^2$ coupling, this is well satisfied, the *l.h.s.* of Eq. (A6) being equal to 1.00021. This probably accounts for the good numerical results of UGT in this case. We find $A = -1.49 \times 10^{-2}$, $B = -1.04 \times 10^{-4}$, independent of the other parameters, and $W^{(0)} = -0.46$ and $W^{(1)} = 0.27$ for $E = a = 1/b = 1$.

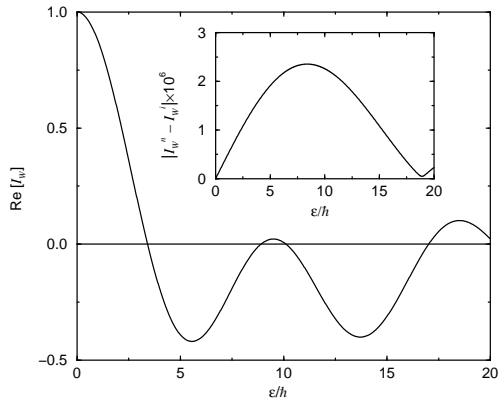


FIG. 2. The real part of I_W as a function of ϵ/\hbar . The difference between this function computed by numerical quadrature (I_W^{num}) and from the interpolation formula (I_W^i) is shown in the inset.

To obtain a fourth parameter, UGT formally take g_E'' as an empirical parameter depending on ϵ , arguing that an independent evaluation of g_E is rather laborious and time consuming. However, g_E is a geometrical quantity

depending on the invariant tori, defined only at $\epsilon = 0$. Its definition cannot be easily extended for $\epsilon > 0$, since the topology of the invariant tori changes under perturbation. We found the simple formula, in the present exercise,

$$g_E(I) = J = \left[\left(\frac{4K}{3\pi} \right)^{4/3} E - \left(\frac{a}{b} \right)^{1/3} I^{4/3} \right]^{3/4}. \quad (\text{A7})$$

UGT do not mention the curvatures \hat{W}''_{\min} , \hat{W}''_{\max} explicitly. These parameters are replaced by the determinants of the monodromy matrix $M_p - 1$, at the stable and unstable orbits [see our Eq. (15)]. These determinants can be expressed in terms of \hat{W}'' and g_E'' . Eliminating thus \hat{W}'' , we obtain the result of UGT in our notation,

$$\begin{aligned} \det(M_u - 1) &= -c\epsilon g_E'' W^{(1)} / [1 - A + 2B]^2, \\ \det(M_s - 1) &= c\epsilon g_E'' W^{(1)} / [1 + A + 2B]^2. \end{aligned} \quad (\text{A8})$$

Here c is a known constant.

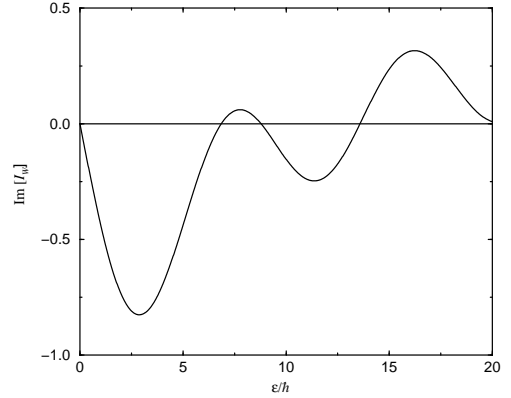


FIG. 3. The imaginary part of I_W as a function of ϵ/\hbar .

Given g_E'' and $W^{(1)}$, we can find A and B in terms of the determinants. Setting $B = 0$ leaves too few parameters, so g_E'' , extended to be a function of ϵ , is pressed into service by UGT. However, for small ϵ , the determinants are proportional to ϵ because the Lyapunov exponents are proportional to $\sqrt{\epsilon}$. Their ratio is a constant not necessarily equal to -1 . The explicit ϵ cancels out leaving $g_E''(\epsilon)$ as nearly constant for small ϵ , but not necessarily close to the zero ϵ limit, unless the system is such that B happens to be small.

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